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Quantum inverse scattering method for multicomponent non-linear Schrödinger model of bosons or fermions with repulsive coupling

Fu-Cho Pu†§, Yi-Zhong Wu‡ and Bao-Heng Zhao‡

† International Centre for Theoretical Physics, Miramare, Trieste, Italy
‡ Graduate School, Chinese Academy of Sciences, Beijing, People's Republic of China

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Abstract. The generalisation of the quantum inverse scattering method to the study of direct and inverse problems for the multicomponent non-linear Schrödinger model of bosons or fermions with repulsive coupling is made. Two sets of Yang-Baxter equations are solved to obtain commutation relations between the scattering state operators. The eigenfunctions have been constructed for the infinite number of conserved quantities and the eigenvalues of the first three conserved quantities—number of particles, momentum and energy—are obtained. The global Izergin-Korepin relations and the relations between the quantum Jost functions are derived, and from them, the quantum Gel'fand-Levitan equations are established. Finally, the series expansion for the field operators in terms of the scattering state operators is written out explicitly.

1. Introduction

Since the advent of the quantum inverse scattering method (QISM), the non-linear Schrödinger model has been extensively studied. It was solved originally for bosons with repulsive coupling independently by Sklyanin and Faddeev (1978), Sklyanin (1979), Thacker and Wilkinson (1979) and Honerkamp *et al* (1979). The quantum Gel'fand-Levitan equation in this case was established by Creamer *et al* (1980) and Grosse (1979). The same model with attractive coupling was solved by Göckler (1981a) and the corresponding Gel'fand-Levitan equation was given by Göckler (1981b) and Smirnov (1982). Extension of this model to include the case of fermions with repulsive coupling was done by Pu and Zhao (1984). Various generalisations have also been made with different emphasis (Kulish 1980, 1985).

The present paper generalises directly the works of Sklyanin (1979) and Pu and Zhao (1984) to the multicomponent non-linear Schrödinger model in the case of repulsive interaction for bosons or fermions. In particle theory this is equivalent to the generalisation to the case of higher spin. We adopt the continuum version of QISM used by Sklyanin (1979), which is more convenient for our purpose.

The Hamiltonian of our model is equal to

$$H = \int \left(\frac{\partial u_i^\dagger}{\partial x} \frac{\partial u_i}{\partial x} - \rho c u_i^\dagger u_j^\dagger u_i u_j \right) \mathrm{d}x \tag{1.1}$$

§ Permanent address: Institute of Physics, Chinese Academy of Sciences, Beijing, People's Republic of China.

where $c>0$ is the coupling constant of repulsion. As a convention, the Latin subscript denotes that the component is $1, \ldots, N$, while later in the text, Greek subscripts denote that the component is $1, \ldots, N, N+1$. Thus the dummy Latin indices mean the sum over $1, \ldots, N$. In order to include both cases for bosons and fermions in an unified single version, we introduce a characteristic index ρ in the formulae throughout the paper and let $\rho=-1$ for the case of bosons and $\rho=1$ for the case of fermions. The commutation relations for the field operators are written in the form of

$$[u_i(x), u_j^\dagger(y)]_\rho = \delta_{ij}\delta(x-y)$$
$$[u_i(x), u_j(y)]_\rho = [u_i^\dagger(x), u_j^\dagger(y)]_\rho = 0 \tag{1.2}$$

where $[\ ,\ ]_\rho$ means the commutator for $\rho=-1$ (bosons) and anticommutator for $\rho=1$ (fermions).

The associated linear operator in QISM is

$$L(x,\lambda) = \mathrm{i}\frac{\lambda}{2}J + \mathrm{i}\sqrt{c}\, u_j(x)E_{j,N+1} - \mathrm{i}\sqrt{c}\, u_j^\dagger(x)E_{N+1,j} \tag{1.3}$$

where $E_{\alpha\beta}$ is a $(N+1)\times(N+1)$ matrix, for which all matrix elements are zero except that the element of αth row and βth column is equal to unity, and

$$J = I_{N+1} - 2E_{N+1,N+1} \tag{1.4}$$

and where I_{N+1} is the $(N+1)\times(N+1)$ unit matrix. The monodromy matrix $T(x,y|\lambda)$ is defined by

$$\frac{\partial}{\partial x}T(x,y|\lambda) = :L(x,\lambda)T(x,y|\lambda):$$
$$T(x,y|\lambda)|_{x=y} = I_{N+1} \tag{1.5}$$

where : : means normal product. As is known (Pu and Zhao 1984) in the case of femions $L(x,\lambda)$ is a supermatrix with $L_{i,j}$, $L_{N+1,N+1}$ as elements of even parity and $L_{N+1,i}$, $L_{i,N+1}$ as elements of odd parity. We redefine the related operations of matrices in appendix 1 including both cases of bosons and fermions and collect some formulae that will be used in this work in appendix 2.

Further, we define

$$T^{(-)}(x,\lambda) = \lim_{y\to-\infty} T(x,y|\lambda)E(y,\lambda)$$
$$T^{(+)}(x,\lambda) = \lim_{y\to\infty} E(-y,\lambda)T(y,x|\lambda)$$
$$T(\lambda) = \lim_{x\to\infty, y\to-\infty} E(-x,\lambda)T(x,y|\lambda)E(y,\lambda) \tag{1.6}$$

where $E(x,\lambda) = \exp(\mathrm{i}\frac{1}{2}\lambda Jx)$.

The Neumann expansions for the matrix elements of $T(x,y|\lambda)$ are obtained in a standard way. From the Neumann expansions we conclude that $T_{i,j}^{(\pm)}(x,\lambda)$, $T_{i,N+1}^{(+)}(x,\lambda)$, $T_{N+1,i}^{(-)}(x,\lambda)$, $T_{i,j}(\lambda)$ are analytic functions of λ in the lower half plane, $T_{N+1,N+1}^{(\pm)}(x,\lambda)$, $T_{N+1,i}^{(+)}(x,\lambda)$, $T_{i,N+1}^{(-)}(x,\lambda)$, $T_{N+1,N+1}(\lambda)$ are analytic in the upper half plane and $T_{N+1,i}(\lambda)$, $T_{i,N+1}(\lambda)$ are defined only for real values of λ.

2. Commutation relations

Using (1.5) and (A2.5) and (A2.6), we can obtain

$$\frac{\partial}{\partial x}\left[T(x,y|\lambda)\underset{\rho}{\otimes}\Gamma(x,y|\mu)\right]=:\Gamma(x|\lambda,\mu)T(x,y|\lambda)\underset{\rho}{\otimes}T(x,y|\mu):$$

where

$$\Gamma(x|\lambda,\mu)=L(x,\lambda)\underset{\rho}{\otimes}I_{N+1}+I_{N+1}\underset{\rho}{\otimes}L(x,\mu)+cE_{i,N+1}\underset{\rho}{\otimes}E_{N+1,i}. \tag{2.1}$$

The solution for the Yang-Baxter equation

$$R_\rho(\lambda,\mu)\Gamma(x|\lambda,\mu)=\Gamma(x|\mu,\lambda)R_\rho(\lambda,\mu) \tag{2.2}$$

is

$$R_\rho(\lambda,\mu)=\frac{\mathrm{i}\rho c}{\lambda-\mu+\mathrm{i}\rho c}I_{N+1}\otimes I_{N+1}+\frac{\lambda-\mu}{\lambda-\mu+\mathrm{i}\rho c}P_\rho \tag{2.3}$$

where

$$P_\rho=E_{\alpha\beta}\otimes E_{\beta\alpha}(-\rho)^{p(\alpha)p(\beta)}$$

and the double Greek letters mean summation over $1,\ldots,N,N+1$ by our convention. From (2.1) and (2.2), it follows that (Sklyanin 1979)

$$R_\rho(\lambda,\mu)T(x,y|\lambda)\underset{\rho}{\otimes}T(x,y|\mu)=T(x,y|\mu)\underset{\rho}{\otimes}T(x,y|\lambda)R_\rho(\lambda,\mu) \tag{2.4}$$

from which we obtain commutation relations for finite volume. The commutation relations for infinite volume are obtained from (2.4) by a careful limiting process (Faddeev 1981)

$$R_\rho^{(+)}(\lambda,\mu)T(\lambda)\underset{\rho}{\otimes}T(\mu)=T(\mu)\underset{\rho}{\otimes}T(\lambda)R_\rho^{(-)}(\lambda,\mu) \tag{2.5}$$

where

$$\begin{aligned}R^{(\pm)}(\lambda,\mu)=&\frac{\mathrm{i}\rho c}{(\lambda-\mu+\mathrm{i}\rho c)(\lambda-\mu)}E_{ii}\otimes E_{jj}+\frac{1}{\lambda-\mu+\mathrm{i}\rho c}E_{\alpha j}\otimes E_{j\alpha}\\&+\frac{\lambda-\mu-\mathrm{i}\rho c}{(\lambda-\mu+\mathrm{i}\varepsilon)^2}E_{j,N+1}\otimes E_{N+1,j}\\&-\frac{\rho(\lambda-\mu-\mathrm{i}c)}{(\lambda-\mu+\mathrm{i}\rho c)(\lambda-\mu)}E_{N+1,N+1}\otimes E_{N+1,N+1}\\&\pm\mathrm{i}\pi\delta(\lambda-\mu)E_{jj}\otimes E_{N+1,N+1}\mp\mathrm{i}\pi\delta(\lambda-\mu)E_{N+1,N+1}\otimes E_{jj}.\end{aligned}$$

Note that the matrices in (2.5) are of $(N+1)^2\times(N+1)^2$ and the rows or columns can be labelled by double indices. Further, we write $a(\lambda)$ for $T_{N+1,N+1}(\lambda)$ and $b_j(\lambda)$ for $T_{N+1,j}(\lambda)$. Comparing the $((N+1,N+1),(N+1,N+1))$, $((N+1,N+1),(N+1,j))$ and $((N+1,N+1),(i,j))$ elements of both sides of (2.5), we obtain respectively

$$a(\lambda)a(\mu)=a(\mu)a(\lambda) \tag{2.6}$$

$$a(\lambda)b_j(\mu)=\frac{\lambda-\mu+\mathrm{i}c}{\lambda-\mu+\mathrm{i}\varepsilon}b_j(\mu)a(\lambda) \tag{2.7}$$

and

$$b_i(\lambda)b_j(\mu) = -\frac{\rho(\lambda-\mu)}{\lambda-\mu-\mathrm{i}c} b_j(\mu)b_i(\lambda) - \frac{\mathrm{i}c}{\lambda-\mu-\mathrm{i}c} b_i(\mu)b_j(\lambda) \tag{2.8}$$

while the last two equations can also be written as

$$a(\lambda)R_j(\mu)^{\dagger} = \frac{\lambda-\mu+\mathrm{i}c}{\lambda-\mu+\mathrm{i}\varepsilon} R_j(\mu)^{\dagger} a(\lambda) \tag{2.9}$$

$$R_i(\lambda)^{\dagger}R_j(\mu)^{\dagger} = \frac{-\rho(\lambda-\mu)}{\lambda-\mu+\mathrm{i}c} R_j(\mu)^{\dagger}R_i(\lambda)^{\dagger} - \frac{\mathrm{i}c}{\lambda-\mu+\mathrm{i}c} R_i(\mu)^{\dagger}R_j(\lambda)^{\dagger} \tag{2.10}$$

by introducing

$$R_j(\lambda)^{\dagger} = \frac{\mathrm{i}}{\sqrt{c}} b_j(\lambda)a(\lambda)^{-1}. \tag{2.11}$$

The above procedure evidently fails to give the commutation relations between scattering state operators and their Hermitian conjugates. The difficulty does not arise in the case of a one-component boson for non-linear Schrödinger model where $T_{11}(\lambda)$ and $T_{22}(\lambda)$ are conjugates, and $T_{12}(\lambda)$, $T_{21}(\lambda)$ are anticonjugates to each other. In order to derive such commutation relations, which are necessary in calculating the Green function and S matrix, we have to consider the solution of another Yang–Baxter equation. Instead of (2.1), we have

$$\begin{aligned} \frac{\partial}{\partial x}\left[T(x,y|\lambda)^{\dagger} \underset{\rho}{\otimes} T^{\rho t}(x,y|\lambda)\right] &= :T(x,y|\lambda)^{\dagger} \underset{\rho}{\otimes} T^{\rho t}(x,y|\mu)\Gamma_1(x|\lambda,\mu): \\ \frac{\partial}{\partial x}\left[T^{\rho t}(x,y|\mu) \underset{\rho}{\otimes} T(x,y|\lambda)^{\dagger}\right] &= :T^{\rho t}(x,y|\mu) \underset{\rho}{\otimes} T(x,y|\lambda)^{\dagger}\Gamma_2(x|\lambda,\mu): \end{aligned} \tag{2.12}$$

where

$$\Gamma_1(x|\lambda,\mu) = I_{N+1} \underset{\rho}{\otimes} L^{\rho t}(x,\mu) + L(x,\lambda)^{\dagger} \underset{\rho}{\otimes} I_{N+1} + cE_{j,N+1} \underset{\rho}{\otimes} E_{j,N+1}$$

$$\Gamma_2(x|\lambda,\mu) = L^{\rho t}(x,\mu) \underset{\rho}{\otimes} I_{N+1} + I_{N+1} \underset{\rho}{\otimes} L(x,\lambda)^{\dagger} - \rho c E_{N+1,j} \underset{\rho}{\otimes} E_{N+1,j}.$$

The solution of the new Yang–Baxter equation

$$\tilde{R}_\rho(\lambda,\mu)\Gamma_1(x|\lambda,\mu) = \Gamma_2(x|\lambda,\mu)\tilde{R}_\rho(\lambda,\mu) \tag{2.13}$$

is

$$\tilde{R}_\rho(\lambda,\mu) = \frac{\mathrm{i}\rho c}{\lambda-\mu-\mathrm{i}\rho c(N-1)} Q_\rho + \frac{\lambda-\mu-\mathrm{i}\rho cN}{\lambda-\mu-\mathrm{i}\rho c(N-1)} P_\rho \tag{2.14}$$

where $Q_\rho = E_{\alpha\beta} \otimes E_{\alpha\beta}(-\rho)^{p(\alpha)+p(\beta)}$. By an analogous procedure, we arrive at the commutation relation

$$\tilde{R}_\rho^{(+)}(\lambda,\mu)T(\lambda)^{\dagger} \underset{\rho}{\otimes} T^{\rho t}(\mu) = T^{\rho t}(\mu) \underset{\rho}{\otimes} T(\lambda)^{\dagger}\tilde{R}_\rho^{(-)}(\lambda,\mu) \tag{2.15}$$

or their equivalents

$$\tilde{R}_\rho^{(+)}(\lambda,\mu)^{-1}T^{\rho t}(\mu) \underset{\rho}{\otimes} T(\lambda)^{\dagger} = T(\lambda)^{\dagger} \underset{\rho}{\otimes} T^{\rho t}(\mu)\tilde{R}_\rho^{(-)}(\lambda,\mu)^{-1}. \tag{2.16}$$

In the formulae

$$\tilde{R}_\rho^{(\pm)}(\lambda,\mu)=\frac{\mathrm{i}\rho c}{\lambda-\mu}E_{ij}\otimes E_{ji} +\frac{\lambda-\mu-\mathrm{i}\rho Nc}{\lambda-\mu}(E_{ij}\otimes E_{ji}+E_{j,N+1}\otimes E_{N+1,j}+E_{N+1,j}\otimes E_{j,N+1}) \mp\pi c\delta(\lambda-\mu)E_{j,N+1}\otimes E_{j,N+1}\pm\pi c\delta(\lambda-\mu)E_{N+1,j}\otimes E_{N+1,j} -\frac{\rho(\lambda-\mu-\mathrm{i}c)(\lambda-\mu-\mathrm{i}\rho Nc)}{(\lambda-\mu+\mathrm{i}\varepsilon)^2}E_{N+1,N+1}\otimes E_{N+1,N+1} \tag{2.17}$$

$$\tilde{R}_\rho^{(\pm)}(\lambda,\mu)^{-1}=\Bigg(\frac{-\mathrm{i}\rho c(\lambda-\mu-\mathrm{i}c)}{(\lambda-\mu+\mathrm{i}\varepsilon)^2}E_{ij}\otimes E_{ji} +\frac{(\lambda-\mu-\mathrm{i}c)}{\lambda-\mu}(E_{ij}\otimes E_{ji}+E_{j,N+1}\otimes E_{N+1,j}+E_{N+1,j}\otimes E_{j,N+1}) \mp\rho\pi c\delta(\lambda-\mu)E_{j,N+1}\otimes E_{j,N+1}\pm\rho\pi c\delta(\lambda-\mu)E_{N+1,j}\otimes E_{N+1,j} -\rho E_{N+1,N+1}\otimes E_{N+1,N+1}\Bigg)\frac{(\lambda-\mu)^2[\lambda-\mu-\mathrm{i}\rho c(N-1)]^2}{(\lambda-\mu-\mathrm{i}c)(\lambda-\mu-\mathrm{i}\rho Nc)}. \tag{2.18}$$

Comparing $((j,i),(N+1,N+1))$ for $j\neq i$, $((N+1,i),(N+1,N+1))$ and $((N+1,N+1),(N+1,N+1))$ elements of both sides of (2.15), we obtain

$$b_i(\lambda)^\dagger b_j(\mu)=-\rho\frac{(\lambda-\mu-\mathrm{i}c)}{\lambda-\mu+\mathrm{i}\varepsilon}b_j(\mu)b_i(\lambda)^\dagger \qquad i\neq j \tag{2.19}$$

$$b_i(\lambda)^\dagger a(\mu)^{-1}=\frac{\lambda-\mu}{\lambda-\mu-\mathrm{i}c}a(\mu)^{-1}b_i(\lambda)^\dagger \tag{2.20}$$

$$a(\lambda)^\dagger a(\mu)=a(\mu)a(\lambda)^\dagger. \tag{2.21}$$

From (2.19) and (2.20), we have

$$R_i(\lambda)R_j(\mu)^\dagger=-\rho\frac{\lambda-\mu}{\lambda-\mu-\mathrm{i}c}R_j(\mu)^\dagger R_i(\lambda) \qquad i\neq j. \tag{2.22}$$

Similarly, from the comparison of $((i,i),(N+1,N+1))$ elements of both sides in (2.15), we also obtain

$$R_i(\lambda)R_i(\mu)^\dagger=-\rho\frac{\lambda-\mu}{\lambda-\mu-\mathrm{i}c}R_i(\mu)^\dagger R_i(\lambda)+\frac{\mathrm{i}c}{\lambda-\mu-\mathrm{i}c}R_j(\mu)^\dagger R_j(\lambda)+2\pi\delta(\lambda-\mu). \tag{2.23}$$

Here $i=1,2,\ldots,N$ and the double indices i do not mean summation. Thus, we have obtained all the necessary commutation relations.

3. Conserved quantities and their eigenstates

The asymptotic expansion of $a(\lambda)$ for large λ can be obtained directly by using the integration by parts to the Neumann expansion for $a(\lambda)$. It takes the form

$$a(\lambda)=1-\frac{c}{i\lambda}N+\frac{1}{(i\lambda)^2}[-icP+\tfrac{1}{2}c^2N(N-1)]+\frac{1}{(i\lambda)^3}$$
$$\times[cH+ic^2(N-1)P-\tfrac{1}{6}c^3N(N-1)(N-2)]+O\left(\frac{1}{\lambda^4}\right) \tag{3.1}$$

where

$$N=\int u_i^\dagger(x)u_i(x)\,dx$$
$$P=-i\int u_i^\dagger(x)\frac{\partial}{\partial x}u_i(x)\,dx$$

are the particle number and momentum operators. Finally, we obtain

$$\ln a(\lambda)=\sum_{n=1}^{\infty}\frac{C_n}{\lambda^n}$$

with

$$C_1=-cN$$
$$C_2=-icP-\tfrac{1}{2}c^2N$$
$$C_3=cH--ic^2P-\tfrac{1}{3}c^3N. \tag{3.2}$$

From (2.6), it follows that all C_i are commuting with each other. Hence, $N, P, H, \ldots$, are conserved quantities.

Define the vacuum state $|0\rangle$ by $u_j(x)|0\rangle=0$. It leads to $R_j(\lambda)|0\rangle=0$. From the commutation relation (2.9), we know that the state

$$|\lambda_1,\lambda_2,\ldots,\lambda_n\rangle=R_{j_1}(\lambda_1)^\dagger R_{j_2}(\lambda_2)^\dagger\ldots R_{j_n}(\lambda_n)^\dagger|0\rangle \tag{3.3}$$

is the eigenstate of $a(\lambda)$ with eigenvalue

$$\Lambda(\lambda;\lambda_1,\ldots,\lambda_n)=\prod_{k=1}^{n}\frac{\lambda-\lambda_k+ic}{\lambda-\lambda_k+i\varepsilon}. \tag{3.4}$$

Therefore, the state is the eigenstate of an infinite number of conserved quantities. For the first three conserved quantities N, P, H, the eigenvalues are n, $\Sigma_{k=1}^n\lambda_k$, $\Sigma_{k=1}^n\lambda_k^2$, respectively.

From (3.4), it is obvious that $a(\lambda)^{-1}$ is analytic in the upper half plane in a weak sense.

4. Izergin–Korepin relations and relations between Jost functions

The quantum analogue of the inverse of a monodromy matrix was firstly established by Izergin and Korepin (1981) through an inversion of $L_n(\lambda)$ in the lattice version. Izergin and Korepin relations have proved to be very useful to derive relations between quantum Jost functions, which form the basis to construct quantum Gel'fand–Levitan equations (Smirnov 1982). Here we give a direct derivation in the continuum version.

Starting from the auxiliary linear (1.5) and (A2.5) and (A2.6) we obtain

$$\frac{\partial}{\partial x}T(x,y|\lambda)^{-1}=:T(x,y|\lambda)^{-1}[\tfrac{1}{2}c+JL(x,\lambda^*-ic)^\dagger J]: \tag{4.1}$$

substituting $T(x, y|\lambda)^{-1} = \exp[c/2(x-y)]S(x, y|\lambda)$ into (4.1), we obtain the differential equation for $S(x, y|\lambda)$

$$\frac{\partial}{\partial x} S(x, y|\lambda) =: S(x, y|\lambda) JL(x, \lambda^* - \mathrm{i}c)^\dagger J:$$

$$S(x, y|\lambda)|_{x=y} = I_{N+1}. \tag{4.2}$$

From the solution of (4.2), we arrive at

$$T(x, y|\lambda)^{-1} = \exp[c(x-y)/2] JT(x, y|\lambda^* - \mathrm{i}c)^\dagger J. \tag{4.3}$$

In a similar way, we obtain

$$\frac{\partial}{\partial x} T^{\rho\prime}(x, y|\lambda)^{-1} =: (JL^{\rho\prime}(x, \lambda^* - \mathrm{i}\rho Nc)^{\dagger\rho} J - \tfrac{1}{2}\rho Nc) T^{\rho\prime}(x, y|\lambda)^{-1}: \tag{4.4}$$

and

$$T^{\rho\prime}(x, y|\lambda)^{-1} = \exp[-\rho Nc(x-y)/2] JT^{\rho\prime}(x, y|\lambda^* - \mathrm{i}\rho Nc)^{\dagger\rho} J. \tag{4.5}$$

Equations (4.3) and (4.4) can be called the global Izergin–Korepin relations. In the lattice version, $L_n(\lambda) = T((n+1)\Delta, n\Delta|\lambda)$ for small lattice spacing Δ, then (4.3) and (4.5) become

$$L_n(\lambda)^{-1} = \exp(c\Delta/2) JL_n(\lambda^* - \mathrm{i}c)^\dagger J \tag{4.6}$$

$$L_n^{\rho\prime}(\lambda)^{-1} = \exp(-\rho Nc\Delta/2) JL_n(\lambda^* - \mathrm{i}\rho Nc)^{\dagger\rho} J \tag{4.7}$$

which are the original form of the Izergin–Korepin relations.

Further, we define Jost functions $\Phi^{(\alpha)}(x, \lambda)$, $\Psi^{(\alpha)}(x, \lambda)$, $\alpha = 1, \ldots, N+1$ as column vectors of $N+1$ components by

$$\Phi^{(\alpha)}(x, \lambda)_\beta = (E(x, \lambda) T^{(+)}(x, \lambda) J)_{\alpha\beta}$$

$$\Psi^{(\alpha)}(x, \lambda^*)_\beta = (T^{(-)}(x, \lambda) E(-x, \lambda))^\dagger_{\alpha\beta}. \tag{4.8}$$

According to the definitions for $T^{(\mp)}(x, \lambda)$ given by (1.6)

$$T^{(-)}(x, \lambda) = \lim_{L\to\infty} T_L^{(-)}(x, \lambda) E(-L, \lambda)$$

$$T^{(+)}(x, \lambda) = \lim_{L\to\infty} E(-L, \lambda) T_L^{(+)}(x, \lambda) \tag{4.9}$$

where

$$T_L^{(-)}(x, \lambda) = T(x, -L|\lambda)$$

$$T_L^{(+)}(x, \lambda) = T(L, x|\lambda).$$

We can also define $\Phi_L^{(\alpha)}(x, \lambda)$, $\Psi_L^{(\alpha)}(x, \lambda)$ in analogy to (4.8)

$$\left.\begin{aligned} &\Phi_L^{(\alpha)}(x, \lambda) = (T_L^{(+)}(x, \lambda) J)_{\alpha\beta} \\ &\Psi_L^{(\alpha)}(x, \lambda) = (T_L^{(-)}(x, \lambda)^\dagger)_{\alpha\beta}. \end{aligned}\right. \tag{4.10}$$

If we write $T_L(\lambda)$ for $T(L, -L|\lambda)$, then

$$\begin{aligned} T_L^{(+)}(x, \lambda - \mathrm{i}c) &= T_L(\lambda - \mathrm{i}c) T_L^{(-)}(x, \lambda - \mathrm{i}c)^{-1} \\ &= \exp[c(x+L)/2] T_L(\lambda - \mathrm{i}c) JT_L^{(-)}(x, \lambda)^\dagger J \end{aligned} \tag{4.11}$$

in which (4.3) is used. Extracting from (4.11) the following relation:

$$T_L(\lambda - \mathrm{i}c)^{-1}_{N+1,N+1}\Phi_L^{(N+1)}(x, \lambda - \mathrm{i}c) = \exp[c(x+L)/2] \\ \times[T_L(\lambda - \mathrm{i}c)^{-1}_{N+1,N+1}T_L(\lambda - \mathrm{i}c)_{N+1,j}\Psi_L^{(j)}(x, \lambda) - \Psi_L^{(N+1)}(x, \lambda)] \tag{4.12}$$

and using the relation

$$T_L(\lambda - \mathrm{i}c)^{-1}_{N+1,N+1}T_L(\lambda - \mathrm{i}c)_{N+1,j} = T_L(\lambda)_{N+1,j}T_L(\lambda)^{-1}_{N+1,N+1}$$

which follows from the comparison of $((N+1, N+1), (N+1, j))$ elements of both sides of (2.4), we obtain

$$a(\lambda - \mathrm{i}c)^{-1}\Phi^{(N+1)}(x, \lambda - \mathrm{i}c) = \exp(-\mathrm{i}\lambda x)b_j(\lambda)a(\lambda)^{-1}\Psi^{(j)}(x, \lambda) - \Psi^{(N+1)}(x, \lambda). \tag{4.13}$$

In the derivation, the following asymptotic behaviours for $\Phi_L^{(\alpha)}(x, \lambda)$, $\Psi_L^{(\alpha)}(x, \lambda)$ are taken into account:

$$\begin{aligned} &\Phi_L^{(j)}(x, \lambda) \sim \exp[\tfrac{1}{2}\mathrm{i}\lambda(L-x)]\Phi^{(j)}(x, \lambda) \\ &\Psi_L^{(j)}(x, \lambda) \sim \exp[-\tfrac{1}{2}\mathrm{i}\lambda(L+x)]\Psi^{(j)}(x, \lambda) \\ &\Phi_L^{(N+1)}(x, \lambda) \sim \exp[-\tfrac{1}{2}\mathrm{i}\lambda(L-x)]\Phi^{(N+1)}(x, \lambda) \\ &\Psi_L^{(N+1)}(x, \lambda) \sim \exp[\tfrac{1}{2}\mathrm{i}\lambda(L+x)]\Psi^{(N+1)}(x, \lambda) \end{aligned} \tag{4.14}$$

when $L \to \infty$.

Equation (4.13) is one of the relations between Jost functions. For the other relation, we start with the second global Izergin–Korepin relation (4.5) and proceed in similar steps as in deriving (4.13). We finally obtain

$$\Phi^{(j)}(x, \lambda - \mathrm{i}\rho Nc) = [\Psi^{(k)}(x, \lambda) - l_\rho\Psi^{(N+1)}(x, \lambda)a(\lambda)^{\dagger-1}b_k(\lambda)^{\dagger}\mathrm{e}^{\mathrm{i}\lambda x}]T(\lambda - \mathrm{i}\rho Nc)_{jk} \tag{4.15}$$

where $l_\rho = -J$ when $\rho = 1$, and $l_\rho = I_{N+1}$ when $\rho = -1$. In the derivation, we have used the relation

$$T_L(\lambda - \mathrm{i}\rho Nc)_{k,N+1} = (T_L(\lambda)^{\dagger}_{N+1,N+1})^{-1}T_L(\lambda)^{\dagger}_{N+1,j}T_L(\lambda - \mathrm{i}\rho Nc)_{kj} \tag{4.16}$$

which comes from the equality of $(N+1, k)$ elements of both sides of

$$\begin{aligned} I_{N+1} &= T_L^{\rho t}(\lambda - \mathrm{i}\rho Nc)^{-1}T_L^{\rho t}(\lambda - \mathrm{i}\rho Nc) \\ &= \exp(-\mathrm{i}\rho Nc)JT_L^{\rho t}(\lambda)^{\dagger\rho}JT_L^{\rho t}(\lambda - \mathrm{i}\rho Nc). \end{aligned} \tag{4.17}$$

5. System of Gel'fand–Levitan equations

From the analyticities for $T^{(+)}(x, \lambda)$, $T^{(-)}(x, \lambda)$, it is easy to see that $\Psi^{(N+1)}(x, \lambda)$ is an analytic function of λ for $\mathrm{Im}\,\lambda < 0$ and $\Psi^{(N+1)}(x, \lambda) = E_{N+1} + \mathrm{O}(1/\lambda)$ for large λ; $\Psi^{(i)}(x, \lambda)$ is analytic for $\mathrm{Im}\,\lambda > 0$ and $\Psi^{(i)}(x, \lambda) = E_i + \mathrm{O}(1/\lambda)$, $i = 1, \ldots, N$. Here E_α denotes a column vector with all components equal to zero except the αth component, which is equal to unity.

Let $G(x, \lambda) = a(\lambda - \mathrm{i}c)^{-1}\Phi^{(N+1)}(x, \lambda - \mathrm{i}c)$, then (5.3) can be written as

$$G(x, \lambda) = \exp(-\mathrm{i}\lambda x)b_j(\lambda)a(\lambda)^{-1}\Psi^{(j)}(x, \lambda) - \Psi^{(N+1)}(x, \lambda). \tag{5.1}$$

From (4.15), we solve for the quantity in the square bracket

$$[\Psi^{(k)}(x, \lambda) - l_\rho\Psi^{(N+1)}(x, \lambda)a(\lambda)^{\dagger-1}b_k(\lambda)^{\dagger}\exp(\mathrm{i}\lambda x)] = F^{(k)}(x, \lambda) \tag{5.2}$$

where $F^{(k)}(x, \lambda)$ is an operator. By examining the partial differential equations satisfied by $G(x, \lambda)$ and $F^{(k)}(x, \lambda)$, and the boundary conditions for them when $x \to \infty$, we conclude that $G(x, \lambda)$ and $F^{(k)}(x, \lambda)$ are analytic functions of λ for $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$, respectively (Pu and Zhao 1986).

If we define

$$f(x, \lambda) = \begin{cases} G(x, \lambda) & \text{when } \operatorname{Im}\lambda > 0 \\ -\Psi^{(N+1)}(x, \lambda) & \text{when } \operatorname{Im}\lambda < 0 \end{cases} \tag{5.3}$$

then, from what has been discussed above, $f(x, \lambda)$ is analytic in both upper and lower half plane, but with a discontinuity across the real axis:

$$\operatorname{disc} f(x, \lambda) = -\mathrm{i}\sqrt{c}\exp(-\mathrm{i}\lambda x) R_j(\lambda)^{\dagger}\Psi^{(j)}(x, \lambda). \tag{5.4}$$

The Cauchy formula leads to the following dispersion relation:

$$\Psi^{(N+1)}(x, \lambda) = E_{N+1} + \frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty} \frac{R_j(\mu)\Psi^{(j)}(x, \mu)\exp(-\mathrm{i}\mu x)}{\mu - \lambda + \mathrm{i}\varepsilon}\,\mathrm{d}\mu \qquad \lambda \in \mathbb{R}. \tag{5.5}$$

This is the first Gel'fand-Levitan equation.

Similarly, if we define

$$h^{(k)}(x, \lambda) = \begin{cases} \Psi^{(k)}(x, \lambda) & \text{when } \operatorname{Im}\lambda > 0 \\ F^{(k)}(x, \lambda) & \text{when } \operatorname{Im}\lambda < 0 \end{cases}$$

$$k = 1, 2, \ldots, N \tag{5.6}$$

then $h^{(k)}(x, \lambda)$ is an analytic function of λ in both upper and lower half plane with a discontinuity across the real axis

$$\operatorname{disc} h^{(k)}(x, \lambda) = l_p \Psi^{(N+1)}(x, \lambda) a(\lambda)^{\dagger -1} b_k(\lambda)^{\dagger}\exp(\mathrm{i}\lambda x). \tag{5.7}$$

The other N Gel'fand-Levitan equations are obtained in a similar fashion

$$\Psi^{(k)}(x, \lambda) = E_k - \frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty} l_p \frac{\Psi^{(N+1)}(x, \mu) R_k(\mu)\exp(\mathrm{i}\mu x)}{\lambda - \mu + \mathrm{i}\varepsilon}\,\mathrm{d}\mu$$

$$\lambda \in \mathbb{R},\ k = 1, 2, \ldots, N. \tag{5.8}$$

Equations (5.5) and (5.8) constitute the complete system of Gel'fand-Levitan equations.

Solving the system of (5.5) and (5.8) by iterations, we obtain

$$\begin{aligned}\Psi^{(k)}_{N+1}(x, \lambda) = \sqrt{c}\sum_{m=0}^{\infty} \frac{c^m(-1)^{m+1}}{(2\pi)^{2m+1}} \int & R_{i_1}(\nu_1)^{\dagger} R_{i_2}(\nu_2)^{\dagger}\cdots R_{i_m}(\nu_m)^{\dagger} \\ & \times R_{i_m}(\mu_m)\cdots R_{i_2}(\mu_2) R_{i_1}(\mu_1) R_k(\mu_0) \\ & \times \frac{\exp[\mathrm{i}\mu x + \mathrm{i}\Sigma_{l=1}^{m}(\mu_l - \nu_l)x]}{\Pi_{j=1}^{m}[(\nu_j - \mu_{j-1} + \mathrm{i}\varepsilon)(\nu_j - \mu_j + \mathrm{i}\varepsilon)](\lambda - \mu + \mathrm{i}\varepsilon)}\,\mathrm{d}\mu_0 \prod_{l=1}^{m}\mathrm{d}\nu_l\,\mathrm{d}\mu_l.\end{aligned} \tag{5.9}$$

From the comparison with the asymptotic expansion of $\Psi^{(k)}_{N+1}(x, \lambda)$,

$$\Psi^{(k)}_{N+1}(x, \lambda) = \frac{-\sqrt{c}}{\lambda} u_k(x) + \mathrm{O}\left(\frac{1}{\lambda^2}\right) \qquad \text{as } |\lambda| \to \infty \tag{5.10}$$

the final solution of the field operator $u_k(x, t)$ in terms of $R_i(\nu)$ can be written in series as

$$u_k(x, t) = \sum_{m=0}^{\infty} \frac{(-c)^m}{(2\pi)^{2m+1}} \int R_{i_1}(\nu_1)^\dagger R_{i_2}(\nu_2)^\dagger \ldots R_{i_m}(\nu_m)^\dagger R_{i_m}(\mu_m) \ldots R_{i_2}(\mu_2)$$

$$\times R_{i_1}(\mu_1) R_k(\mu_0) \prod_{j=1}^{m} (\nu_j - \mu_{j-1} + \mathrm{i}\varepsilon)^{-1} (\nu_j - \mu_j + \mathrm{i}\varepsilon)^{-1} \exp(\mathrm{i}\mu_0 x - \mathrm{i}\mu_0^2 t)$$

$$\times \prod_{l=1}^{m} \{\exp[\mathrm{i}x(\mu_l - \nu_l) - \mathrm{i}t(\mu_l^2 - \nu_l^2)]\, \mathrm{d}\mu_l\, \mathrm{d}\nu_l\}\, \mathrm{d}\mu_0. \tag{5.11}$$

Note that in (5.11), we have substituted $R_j(\nu_i)^\dagger \exp(\mathrm{i}\lambda^2 t)$ for $R_j(\nu_i)^\dagger$ to express the field operator as a function of both coordinate x and time t.

It should be noted that the results in direct problem (3.2)-(3.4) and the inverse problem (5.11) are the same for both cases of bosons and fermions, but the commutation relations between $R_i(\lambda)$ and $R_j(\mu)$, $R_j(\mu)^\dagger$ ((2.10), (2.22) and (2.23)) are different.

Having found the inversion formula (5.11) and the commutation relations between $R_i(\lambda)$, $R_j(\lambda)^\dagger$, it is straightforward to calculate the Green function and S matrix.

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Appendix 1

The supermatrices in $M_{p,q}(A)$ should be used in the case of fermions. However, in the case of bosons only ordinary matrices are needed. In order to unify the treatment for both cases, we define

$$X^{\rho t} = \begin{cases} X^t & \text{when } \rho = -1 \\ X^{st} & \text{when } \rho = 1 \end{cases} \tag{A1.1}$$

$$\underset{\rho}{\otimes} = \begin{cases} \otimes & \text{when } \rho = -1 \\ \underset{s}{\otimes} & \text{when } \rho = 1 \end{cases} \tag{A1.2}$$

where $\rho = -1$ is for bosons and $\rho = 1$ is for fermions. Thus, we have accordingly

$$(X^{\rho t})_{\alpha\beta} = X_{\beta\alpha}(-\rho)^{p(\alpha)(p(\beta)+1)} \tag{A1.3}$$

$$(XY)^{\rho t} = Y^{\rho t} X^{\rho t} \tag{A1.4}$$

$$\left(X \underset{\rho}{\otimes} Y\right)_{\alpha\beta,\gamma\delta} = X_{\alpha\gamma} Y_{\beta\delta}(-\rho)^{p(\beta)(p(\alpha)+p(\gamma))} \tag{A1.5}$$

$$\left(X_1 \underset{\rho}{\otimes} Y_1\right)\left(X_2 \underset{\rho}{\otimes} Y_2\right) = X_1 X_2 \underset{\rho}{\otimes} Y_1 Y_2. \tag{A1.6}$$

For matrix X, we define another matrix X^ρ by

$$(X^\rho)_{\alpha\beta} = (-\rho)^{p(\alpha)+p(\beta)} X_{\alpha\beta}. \tag{A1.7}$$

The following formulae used in the paper are easy to prove for ordinary matrices A, B, D:

$$\left(A \underset{\rho}{\otimes} E_{\alpha\beta}\right)\left(B \underset{\rho}{\otimes} D\right) = A\hat{B} \underset{\rho}{\otimes} E_{\alpha\beta}D$$

$$\left(A \underset{\rho}{\otimes} B\right)\left(E_{\alpha\beta} \underset{\rho}{\otimes} D\right) = AE_{\alpha\beta} \underset{\rho}{\otimes} \hat{B}D \tag{A1.8}$$

where

$$\hat{B}_{\lambda\mu} = B_{\lambda\mu}(-\rho)^{(p(\alpha)+p(\beta))(p(\lambda)+p(\mu))}$$

$$\exp\left(A \underset{\rho}{\otimes} I + I \underset{\rho}{\otimes} B\right) = (\exp A) \underset{\rho}{\otimes} (\exp B). \tag{A1.9}$$

Appendix 2

In this appendix, we collect some formulae used in the paper for convenience:

$$\frac{\partial}{\partial x} T(x, y|\lambda)^{\dagger} = :T(x, y|\lambda)^{\dagger} L(x, \lambda)^{\dagger}:$$

$$\frac{\partial}{\partial y} T(x, y|\lambda)^{\dagger} = -:L(y, \lambda)^{\dagger} T(x, y|\lambda)^{\dagger}: \tag{A2.1}$$

where

$$L(x, \lambda)^{\dagger} = -\tfrac{1}{2}\mathrm{i}\lambda J - \mathrm{i}\sqrt{c}\,u_j^{\dagger}(x)E_{N+1,j} + \mathrm{i}\sqrt{c}\,u_j(x)E_{j,N+1}$$

$$\frac{\partial}{\partial x} T^{\rho t}(x, y|\lambda) = :T^{\rho t}(x, y|\lambda) L^{\rho t}(x, y):$$

$$\frac{\partial}{\partial y} T^{\rho t}(x, y|\lambda) = -:L^{\rho t}(y, \lambda) T^{\rho t}(x, y|\lambda): \tag{A2.2}$$

where

$$L^{\rho t}(x, \lambda) = \tfrac{1}{2}\mathrm{i}\lambda J - \mathrm{i}\sqrt{c}\rho u_j(x)E_{N+1,j} - \mathrm{i}\sqrt{c}\,u_j^{\dagger}(x)E_{j,N+1}$$

$$:u_j(z)T(x, y|\lambda): = :T(x, y|\lambda)^{\rho} u_j(z):$$

$$:T(x, y|\lambda)u_j^{\dagger}(z): = :u_j^{\dagger}(z)T(x, y|\lambda)^{\rho}: \tag{A2.3}$$

$$u_j(z)T(x, y|\lambda) = T(x, y|\lambda)^{\rho} u_j(z)$$

$$T(x, y|\lambda)u_j^{\dagger}(z) = u_j^{\dagger}(z)T(x, y|\lambda)^{\rho} \qquad \forall z \notin [x, y] \tag{A2.4}$$

$$u_j(x)T(x, y|\lambda) = T(x, y|\lambda)^{\rho} u_j(x) - \tfrac{1}{2}\mathrm{i}\sqrt{c}E_{N+1,j}T(x, y|\lambda)$$

$$T(x, y|\lambda)u_j^{\dagger}(x) = u_j^{\dagger}(x)T(x, y|\lambda)^{\rho} + \tfrac{1}{2}\mathrm{i}\sqrt{c}E_{j,N+1}T(x, y|\lambda)^{\rho} \tag{A2.5}$$

$$u_j(y)T(x, y|\lambda) = T(x, y|\lambda)^{\rho} u_j(y) - \tfrac{1}{2}\mathrm{i}\sqrt{c}T(x, y|\lambda)^{\rho}E_{N+1,j}$$

$$T(x, y|\lambda)u_j^{\dagger}(y) = u_j^{\dagger}(y)T(x, y|\lambda)^{\rho} + \tfrac{1}{2}\mathrm{i}\sqrt{c}T(x, y|\lambda)E_{j,N+1} \tag{A2.6}$$

$$u_j(x)T(x,y|\lambda)^\dagger = T(x,y|\lambda)^{\dagger\rho}u_j(x) - \tfrac{1}{2}\mathrm{i}\sqrt{c}\,T(x,y|\lambda)^{\dagger\rho}E_{N+1,j}$$
$$T(x,y|\lambda)^\dagger u_j^\dagger(x) = u_j^\dagger(x)T(x,y|\lambda)^{\dagger\rho} + \tfrac{1}{2}\mathrm{i}\sqrt{c}\,T(x,y|\lambda)^\dagger E_{j,N+1} \tag{A2.7}$$

$$u_j(y)T(x,y|\lambda)^\dagger = T(x,y|\lambda)^{\dagger\rho}u_j(y) - \tfrac{1}{2}\mathrm{i}\sqrt{c}\,E_{N+1,j}T(x,y|\lambda)^\dagger$$
$$T(x,y|\lambda)^\dagger u_j^\dagger(y) = u_j^\dagger(y)T(x,y|\lambda)^{\dagger\rho} + \tfrac{1}{2}\mathrm{i}\sqrt{c}\,E_{j,N+1}T(x,y|\lambda)^{\dagger\rho} \tag{A2.8}$$

$$u_j(x)T^{\rho\prime}(x,y|\lambda) = T^{\rho\prime}(x,y|\lambda)^\rho u_j(x) - \tfrac{1}{2}\mathrm{i}\sqrt{c}\,T^{\rho\prime}(x,y|\lambda)^\rho E_{j,N+1}$$
$$T^{\rho\prime}(x,y|\lambda)u_j^\dagger(x) = u_j^\dagger(x)T^{\rho\prime}(x,y|\lambda)^\rho - \rho\tfrac{1}{2}\mathrm{i}\sqrt{c}\,T^{\rho\prime}(x,y|\lambda)E_{N+1,j} \tag{A2.9}$$

$$u_j(y)T^{\rho\prime}(x,y|\lambda) = T^{\rho\prime}(x,y|\lambda)^\rho u_j(y) - \tfrac{1}{2}\mathrm{i}\sqrt{c}\,E_{j,N+1}T^{\rho\prime}(x,y|\lambda)$$
$$T^{\rho\prime}(x,y|\lambda)u_j^\dagger(y) = u_j^\dagger(y)T^{\rho\prime}(x,y|\lambda) - \rho\tfrac{1}{2}\mathrm{i}\sqrt{c}\,E_{N+1,j}T^{\rho\prime}(x,y|\lambda)^\rho \tag{A2.10}$$

$$u_j(x)T^{(-)}(x,\lambda) = T^{(-)}(x,\lambda)^\rho u_j(x) - \tfrac{1}{2}\mathrm{i}\sqrt{c}\,E_{N+1,j}T^{(-)}(x,\lambda)$$
$$T^{(-)}(x,\lambda)u_j^\dagger(x) = u_j^\dagger(x)T^{(-)}(x,\lambda)^\rho + \tfrac{1}{2}\mathrm{i}\sqrt{c}\,E_{j,N+1}T^{(-)}(x,\lambda)^\rho \tag{A2.11}$$

$$u_j(x)T^{(+)}(x,\lambda) = T^{(+)}(x,\lambda)^\rho u_j(x) - \tfrac{1}{2}\mathrm{i}\sqrt{c}\,T^{(+)}(x,\lambda)^\rho E_{N+1,j}$$
$$T^{(+)}(x,\lambda)u_j^\dagger(x) = u_j^\dagger(x)T^{(+)}(x,\lambda)^\rho + \tfrac{1}{2}\mathrm{i}\sqrt{c}\,T^{(+)}(x,\lambda)E_{j,N+1} \tag{A2.12}$$

$$u_j(x)T(\lambda) = T(\lambda)^\rho u_j(x) - \mathrm{i}\sqrt{c}\,T^{(+)}(x,\lambda)^\rho E_{N+1,j}T^{(-)}(x,\lambda)$$
$$T(\lambda)u_j^\dagger(x) = u_j^\dagger(x)T(\lambda)^\rho + \mathrm{i}\sqrt{c}\,T^{(+)}(x,\lambda)E_{j,N+1}T^{(-)}(x,\lambda)^\rho. \tag{A2.13}$$

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